x-27/2071 5953/ML

PHD 02: Topology and Algebra

Max. Marks: 50 **Time Allowed: 3 hours**

Instructions for Candidates: 1. Candidates are required to attempt one question each from sections A, B, C and D and the entire section E.

2. Use of non programmable scientific calculator is allowed.

Section-A

1. (i) Let $\{Y_\alpha : \alpha \in A\}$ be any family of topological spaces. Also let $U_\alpha \neq Y_\alpha$ be open set in Y_α. Prove that $\prod_\alpha U_\alpha$ is not an open set in $\prod_\alpha Y_\alpha$. 5

   (ii) Let $A_i, i \in \mathbb{Z}^+$ are connected and $A_1 \supset A_2 \supset \cdots$. Prove that $\bigcap_n A_n$ need not be connected. 5

2. (i) Prove that infinite product of discrete spaces is always totally disconnected. 5

   (ii) Let $Y = \left\{(x, y) : y = \sin\frac{1}{x}, 0 < x \leq 1\right\}$. Prove that closure of Y is not path-connected. 5

Section-B

3. (i) Prove that a separable metric space is Lindelof. 5

   (ii) Let $X = \mathbb{R}$ be a space with half ray topology and $Y = [0,1]$ be the space with usual topology. Prove that X and Y are normal spaces but $X \times Y$ is not a normal space. 5

4. (i) Let X be a completely regular, and $C(X)$ be the set of all bounded continuous real valued functions on X. For each $x, f, \in$, let $U(x, f, \in) = \{y : |f(x) - f(y)| < \in\}$. Prove that $\{U(x, f, \in) : for\ all\ f, x, \in > 0\}$ is a sub-bases for the topology of X. 5

   (ii) Prove that the necessary and sufficient condition for a filter f converges to a in X is that there exist a filter g finer than f converging to a. 5

Section-C

5. Let $G = S_n, n \geq 3$ and fix $i \in \{1,2, \ldots, n\}$. Let $G_3 = stab(3)$. Use group action to prove that G_3 is a subgroup of G. Find $|G_3|$. Also find $Orb(i)$ and $|Orb(i)|$ for $i \in \{1,2, \ldots, n\}$. 10

Contd—2

6. (i) Let p be a prime and let G be a group of order p^{α}. Prove that G has a subgroup of order p^{β}, for every $\beta, 0 \leq \beta \leq \alpha$.

5

(ii) Find the number of groups of order p^3, p prime, in which every non-identity element is of order p.

5

Section-D

7. (i) If M is a maximal ideal of non-commutative ring R with unity. Then is R/M division ring? Justify.

5

(ii) Let $R = \{f \mid f: [0,1] \to \mathbb{R},\ f \text{continuous}\}$. Let $M_c = \{f \in R \mid f(c) = 0,$ $c \in [0,1]$. Prove that M_c is a maximal ideal of R and if M is any maximal ideal of R then there exist $c \in [0,1]$ such that $M = M_c$.

5

8. Define fundamental theorem for finite Abelian groups. Does it imply the converse of Lagrange's theorem in general? If not, Explain for which groups the above statement is true(with proof).

10

Section-E

9. (i) Find $Aut(\mathbb{Z}_{11937})$.

2

(ii) Find a prime and maximal ideal (each) of ring $\mathbb{Z}_{100}[x]$.

2

(iii) Find all groups (up to isomorphism) of order 69.

2

(iv) Give an example of a normal non-regular space.

2

(v) Prove that $\prod A_{\alpha}$ is dense in $\prod Y_{\alpha}$ if and only if each A_{α} is dense in Y_{α} where $A_{\alpha} \subset Y_{\alpha}$.

5957/ML

2